RPE
JUSTIN YIRKA

On

Complexity Classification of Product State Problems for Local Hamiltonians

John Kallaugher, Ojas Parekh, Kevin Thompson, Yipu Wang, and Justin Yirka

arXiv: 2401.06725, January 2024

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The University of Texas at Austin

Hamiltonians

An n-qubit **Hamiltonian** is a $2^n \times 2^n$ Hermitian matrix.

It encodes constraints, interactions, or “rules” of a physical system, its eigenvectors correspond to physical states of the system, and its eigenvalues are the energies of those states.

$$\langle\psi|H|\psi\rangle = \lambda$$

$$n\text{-qubit state} \leftrightarrow \text{Vector in } \mathbb{C}^{2^n} \text{ with } \|v\|_2 = 1$$

$$\text{State } |v\rangle \leftrightarrow \text{Vector } v$$

$$\langle v| \leftrightarrow \text{Row vector } v^{T^*}$$

$$\text{State } \rho \leftrightarrow \text{Outer product } |v\rangle\langle v| = v_i^{T^*} v_i$$

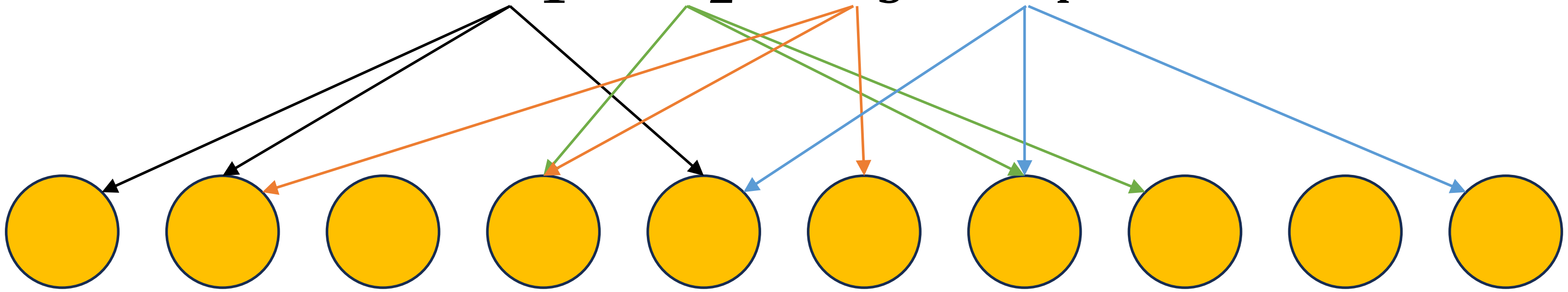
Tensor product $\otimes \leftrightarrow$ “Multiplying” spaces i.e. $\mathbb{R}^4 \otimes \mathbb{R}^4 = \mathbb{R}^{16}$

Local Hamiltonians

A $\boldsymbol{k}$**-local Hamiltonian** is a sum of Hamiltonian terms each acting on at most k qubits

$$H = \sum_i H_{S_i} \otimes \mathbb{I}_{\overline{S_i}} \qquad |S_i| \leq k$$

$$H = H_1 + H_2 + H_3 + H_4 + \cdots$$



k-LH

The ***k*-LH problem** is, given a k-local Hamiltonian, estimate its minimum eigenvalue, a.k.a. its **ground state energy**.

Formally: decide if $\lambda_{\min} < a$ (YES) or $\lambda_{\min} > b$ (NO) for $b - a \geq \frac{1}{\mathrm{poly}(n)}$.

Hamiltonian	Boolean formula
k-local Hamiltonian	k-CNF
k-LH	k-Max-SAT
$\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & 1 \end{bmatrix}_{1,2} + \begin{bmatrix} 0 & & -1 & \\ & 1 & -1 & \\ 1 & & 0 & 1 \\ & & & 1 \end{bmatrix}_{1,3} + \cdots$	$(x_1 \vee x_2) \wedge (x_1 \vee \overline{x_3}) \wedge (x_3 \vee x_4) \wedge \cdots$

Complexity

k-Max-SAT is NP-complete for $k \geq 2$

Similarly, k-LH is QMA-complete for $k \geq 2$

Hardness is subtle, though.

Goal: Characterize the complexity of k**-LH** when the problem is restricted to various subsets of Hamiltonians

Complexity

Just enumerating sets of Hamiltonians seems…

- tedious
- uninformative – what's the underlying structure?
- difficult

So we consider sets of families defined by interesting properties

S-LH

Let a Hamiltonian **family** be defined by the allowed interactions, i.e. by the allowed k-qubit terms.

For a fixed set S of allowed terms/allowed interactions,
the $\boldsymbol{S}$**-LH problem** is k-LH with the additional promise/restriction that any input is of the form

$$H = \sum_i w_i H_i \qquad \text{with each } H_i \in S$$

(we will focus on sets S of 2-qubit, 4×4, terms)

S-LH

For a fixed set S of allowed terms/allowed interactions,
the ***S*-LH problem** is k-LH with the additional promise/restriction that any input is of the form

$$H = \sum_i w_i H_i \qquad \text{with each } H_i \in S$$

- S-LH with $S = \{X \otimes X + Y \otimes Y + Z \otimes Z\}$ is the Quantum Max-Cut problem.
- Classically, $\{\neq\}$-Max-SAT is Max-Cut.
- {2-Out-of-4}-SAT is NP-complete,
  used in *The Power of Unentanglement* [ABDSF 08]

Complexity classification

We're interested in classifying the complexity of S-LH *as a function of* the set S of allowed terms.

Detour:

Complexity classifications of constraint satisfaction problems (CSPs) as a function of the allowed constraints

Complexity classification of CSPs

The complexity of satisfiability problems, Schaefer, 1976

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Schaefer's dichotomy theorem

1 language

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In computational complexity theory, a branch of computer science, **Schaefer's dichotomy theorem**, proved by Thomas Jerome Schaefer, states necessary and sufficient conditions under which a finite set *S* of relations over the Boolean domain yields polynomial-time or NP-complete problems when the relations of *S* are used to constrain some of the propositional variables.[1] It is called a **dichotomy theorem** because the complexity of the problem defined by *S* is either in P or is NP-complete, as opposed to one of the classes of intermediate complexity that is known to exist (assuming P ≠ NP) by Ladner's theorem.

Special cases of Schaefer's dichotomy theorem include the NP-completeness of SAT (the Boolean satisfiability problem) and its two popular variants 1-in-3 SAT and not-all-equal 3SAT (often denoted by NAE-3SAT). In fact, for these two variants of SAT, Schaefer's dichotomy theorem shows that their monotone versions (where negations of variables are not allowed) are also NP-complete.

Original presentation [edit]

Schaefer defines a decision problem that he calls the Generalized Satisfiability problem for *S* (denoted by SAT(*S*)), where $S = \{R_1, \ldots, R_m\}$ is a finite set of relations over the binary domain $\{0, 1\}$. An instance of the problem is an *S*-formula, i.e. a conjunction of constraints of the form $R_j(x_{i_1}, \ldots, x_{i_n})$

Complexity classification of CSPs

Schaefer's dichotomy theorem, 1976:

Given any fixed set S of allowed Boolean constraints,

deciding satisfiability of a formula

$$f(x_1, x_2, \ldots, x_n) = \bigwedge_i C_i \text{ for } C_i \in S$$

a.k.a. ***S*-SAT**

- is in P if any of some simple conditions is true,
- and otherwise is NP-complete.

a) " Every relation in S is 0-valid
b) Every relation in S is 1-valid
c) Every relation in S is weakly positive
d) Every relation in S is weakly negative
e) Every relation in S is affine
f) Every relation in S is bijunctive "

Complexity classification of CSPs

- [Schaefer 1976] classifies S-SAT
- [Creignou 95] with [Khanna, Sudan, Williamson 97] classify S-Max-SAT
- [Jonsson 00] classifies S-Max-SAT with positive & negative weights.
- [Jonsson, Klasson, Krokhin 06] and [Thapper, Živný **16**] extends this to non-binary variables.
  - (Only positive weights)

End of detour, back to quantum

Complexity classification of S-LH

[Cubitt, Montanaro 13] classify S-LH for all sets S of 2-qubit terms:

Given a fixed set S of 2-qubit Hamiltonian terms,
S-LH is either in P, or is NP-, StoqMA-, or QMA-complete.

- If every matrix in S is 1-local, S-LH is in P;
- Otherwise, if there exists $U \in SU(2)$ such that U locally diagonalises S, then S-LH is NP-complete;
- Otherwise, if there exists $U \in SU(2)$ such that, for each 2-qubit matrix
$$H_i \in S, U^{\otimes 2} H_i U^{\dagger \otimes 2} = \alpha_i Z^{\otimes 2} + A_i \otimes I + I \otimes B_i$$
where $\alpha_i \in \mathbb{R}$ and A_i, B_i are arbitrary single-qubit Hermitian matrices, then S-LH is StoqMA-complete; ← [Bravyi, Hastings 2014]
- Otherwise, S-LH is QMA-complete.

What about product states?

What about product states?

We have a full classification of S-LH for 2-qubit terms, i.e. estimating the minimum eigenvalue.

What about other Hamiltonian problems?

- Other ground state properties
- Constrained optimization
- Thermal limit
- …
- Product states

Product states

A **product state** is an unentangled tensor product of single-qubit states

$$\rho = \rho_1 \otimes \rho_2 \otimes \rho_3 \otimes \cdots \otimes \rho_n$$

- Product states can be described efficiently classically.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} e & f \\ g & h \end{bmatrix} \otimes \begin{bmatrix} i & j \\ k & l \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

Product states

A **product state** is an unentangled tensor product of single-qubit states

$$\rho = \rho_1 \otimes \rho_2 \otimes \rho_3 \otimes \cdots \otimes \rho_n$$

- Product states can be described efficiently classically.
- They're intermediate between classical and general quantum states.
- For many natural sets of Hamiltonians, product states are rigorously near-optimal.
  - [Brandao, Harrow 13]: "High"-degree Hamiltonians' ground states are close to product states (monogamy of entanglement!)
- They're a popular ansatz in classical Hamiltonian approximation algorithms

Product state problems

A **product state** is an unentangled tensor product of single-qubit states

$$\rho = \rho_1 \otimes \rho_2 \otimes \rho_3 \otimes \cdots \otimes \rho_n$$

k-LH → **prodLH**

given a local Hamiltonian, estimate the minimum energy over all product states:

$$\min_{\rho=\rho_1\rho_2\ldots\rho_n} \mathrm{Tr}(H\rho)$$

S-LH → ***S*-prodLH**

the problem prodLH restricted to $H = \sum_i w_i H_i$ with $H_i \in S$.

Main Theorem

Drum roll...

Complexity classification of product state problems

<u>Main Theorem</u>

For any fixed set of 2-qubit Hamiltonian terms S,

- ➢ if every matrix in S is 1-local then S-prodLH is in P,
- ➢ and otherwise S-prodLH is NP-complete.

<u>Corollary</u>

For any fixed set of 2-qubit Hamiltonian terms S,

- the problem S-LH is at least NP-hard iff S-prodLH is NP-complete.
- the problem S-LH is in P iff S-prodLH is in P.

Proof outline

Main Theorem

For any fixed set of 2-qubit Hamiltonian terms S,

- if every matrix in S is 1-local then S-prodLH is in P,
- and otherwise S-prodLH is NP-complete.

✓ If every term is 1-local, then we can optimize the state of each qubit individually, so the problem is in P.

✓ prodLH is always contained in NP, using product states' concise classical descriptions $\mathrm{Tr}(H\rho) = \sum_{ij} \mathrm{Tr}(H_{ij}\, \rho_i \rho_j)$

❑ **To Do:** show if S contains a nontrivial 2-qubit term, then S-prodLH is NP-hard.

- Design Hamiltonian gadgets to embed "nice" objective into optimal product state.
- That objective defines a variant of Vector Max-Cut, which we show is NP-complete.

Questions?

Analyzing product state energies

As a warmup, consider the example 2-qubit term

$$H = X \otimes X + Y \otimes Y + Z \otimes Z$$

where X, Y, Z are the Pauli matrices.

$$\left\{ X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \mathrm{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a basis for 2×2 Hermitian matrices

Analyzing product state energies

As a warmup, consider the example 2-qubit term

$$H = XX + YY + ZZ$$

where X, Y, Z are the Pauli matrices.

Write states using **Bloch vectors**:

$$\rho^a = \frac{1}{2}(I + a_1 X + a_2 Y + a_3 Z), \qquad \hat{a} \in \mathbb{R}^3, |\hat{a}| = 1$$

Then

$$\mathrm{Tr}\left(H\, \rho^a \rho^b\right) = \frac{1}{4} \sum_{ij} a_i b_j\, \mathrm{Tr}\left[H\, \sigma_i \sigma_j\right] \qquad \text{for } \sigma_i \in \{X, Y, Z, I\}$$

Cross terms disappear!

$\mathrm{Tr}\left[H \sigma_i \sigma_j\right] \neq 0$ iff $\mathrm{Tr}\left[H \sigma_i \sigma_j\right] = \mathrm{Tr}[II]$, which requires $i = j$

Analyzing product state energies

As a warmup, consider the example 2-qubit term

$$H = XX + YY + ZZ$$

where X, Y, Z are the Pauli matrices.

Write states using **Bloch vectors**:

$$\rho^a = \frac{1}{2}(I + a_1 X + a_2 Y + a_3 Z) \qquad \hat{a} \in \mathbb{R}^3, |\hat{a}| = 1$$

Cross terms disappear!

$$\mathrm{Tr}\left(H \;\rho^a \rho^b\right) = \frac{1}{4}\mathrm{Tr}[a_1 b_1 XX \cdot XX + a_2 b_2 YY \cdot YY + a_3 b_3 ZZ \cdot ZZ]$$

$$= a_1 b_1 + a_2 b_2 + a_3 b_3 = \hat{\boldsymbol{a}} \cdot \hat{\boldsymbol{b}}$$

Analyzing product state energies

$$H = XX + YY + ZZ$$
$$\mathrm{Tr}\left(H\ \rho^a \rho^b\right) = \hat{\boldsymbol{a}} \cdot \hat{\boldsymbol{b}}$$

Product state problems become optimization problems over Bloch vectors!

$\{H\}$-prodLH is equivalent to optimizing sums of inner products:

$$\sum_{uv \in E} w_{uv}\, u \cdot v$$

over unit vectors $u, v \in \mathbb{R}^3$

General product state energies

Write an arbitrary 2-qubit term in the Pauli basis:

$$H = \sum_{i,j=1}^{3} M_{ij}\sigma_i \otimes \sigma_j + \sum_{k=1}^{3} c_k\sigma_k \otimes I + w_k I \otimes \sigma_k$$

3×3 matrix M vectors $\hat{c}$ $\hat{w}$

Then

$$\mathrm{Tr}(H\ \rho^u\rho^v) = \hat{u}^T \mathrm{M}\hat{v} + \hat{u}^T\hat{c} + \hat{v}^T\hat{w}$$

General product state energies

$$\mathrm{Tr}(H\ \ \rho^u \rho^v) = \hat{u}^T \mathrm{M} \hat{v} + \hat{u}^T \hat{c} + \hat{v}^T \widehat{w}$$

For a general 2-qubit H, we still have a sum of inner products,

but with extra terms

and warped by extra coefficients

Can we make this “nicer”?

Hamiltonian gadgets

Trick 1: Symmetrize

In general, the orientation of interactions matters: $H^{ab} \neq H^{ba}$.

It eases analysis if the term is symmetric.

From now on, if we apply H to qubits a, b, we apply it in both directions:

$$H_{sym} = H^{ab} + H^{ba} = H^{ab} + \text{SWAP}\, H^{ab}\, \text{SWAP}$$

Since H_{sym} is symmetric, M' is symmetric, and we can analyze assuming

$$M' = \begin{bmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{bmatrix}$$

$$H_{sym} = \sum_{i,j=1}^{3} M'_{ij} \sigma_i \otimes \sigma_j + \sum_{k=1}^{3} (c_k + w_k)(\sigma_k \otimes I + I \otimes \sigma_k)$$

Trick 2: Delete 1-local terms

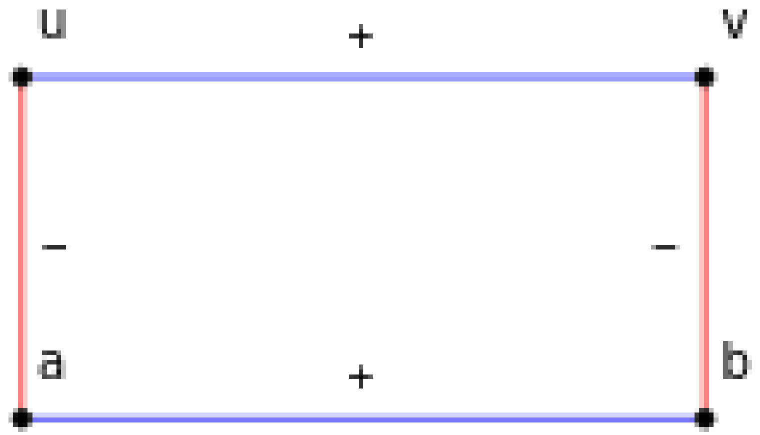
$$H_{sym} = \sum_{i=1}^{3} M'_{ii}\sigma_i \otimes \sigma_i + \sum_{k=1}^{3} (c_k + w_k)(\sigma_k \otimes I + I \otimes \sigma_k)$$

$$\mathrm{Tr}(H\ \rho^u \rho^v) = \hat{u}^T M' \hat{v} + (\hat{c} + \widehat{w})^T (\hat{u} + \hat{v})$$

We borrow a nice gadget from [CM14].

To interact two qubits u, v, we add two ancilla qubits a, b:

$$G^{uv} = H_{sym}^{uv} + H_{sym}^{ab} - H_{sym}^{ua} - H_{sym}^{vb}$$



Negative weights cause all the 1-local terms to cancel

Result of Trick 2

$$H_{sym} = \sum_{i=1}^{3} M'_{ii}\sigma_i \otimes \sigma_i + \sum_{k=1}^{3} (c_k + w_k)(\sigma_k \otimes I + I \otimes \sigma_k)$$

To interact two qubits u, v, we add two ancilla qubits a, b:

$$G^{uv} = H^{uv}_{sym} + H^{ab}_{sym} - H^{ua}_{sym} - H^{vb}_{sym}$$

$$= \cdots \cdots \cdots$$

Then,

$$\mathrm{Tr}[G^{uv}\,\rho_u\rho_v\rho_a\rho_b] = (\hat{\boldsymbol{u}} - \hat{\boldsymbol{v}})^{\boldsymbol{T}}\boldsymbol{M}'(\hat{\boldsymbol{a}} - \hat{\boldsymbol{b}})$$

Bloch vectors

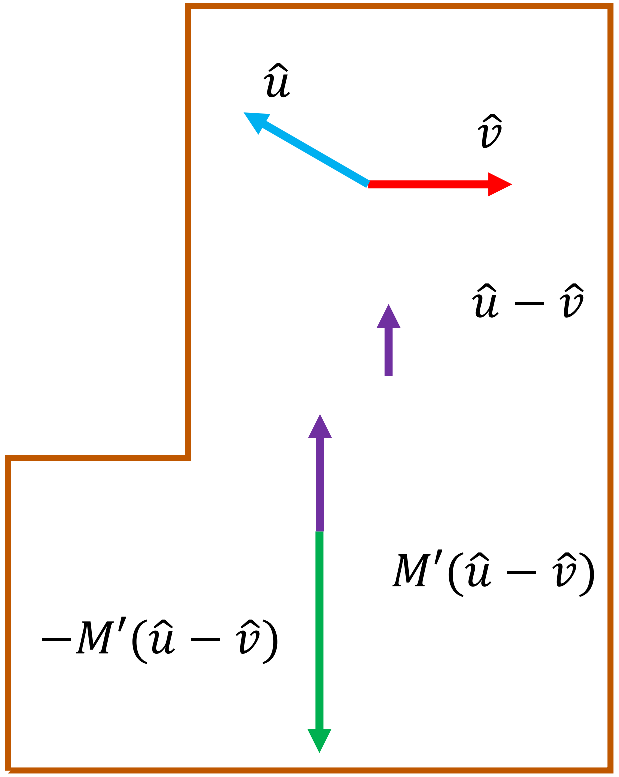
$$M' = \begin{bmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{bmatrix}$$

Result of tricks

$$H_{sym} = \sum_{i=1}^{3} M'_{ii}\sigma_i \otimes \sigma_i + \sum_{k=1}^{3} (c_k + w_k)(\sigma_k \otimes I + I \otimes \sigma_k)$$

To interact two qubits u, v, we add two ancilla qubits a, b, and construct gadget G.

$\mathrm{Tr}[G^{uv}\ \rho_u\rho_v\rho_a\rho_b]\ = (\hat{u} - \hat{v})^T M'(\hat{a} - \hat{b})$



Given u, v are constrained, what is the minimum value of $(\hat{u} - \hat{v})^T M'(\hat{a} - \hat{b})$?

Qubits a, b are free, each become proportional to $-M'(\hat{u} - \hat{v})$,

So minimum value is… $-2(\hat{u} - \hat{v})^T M'' \frac{M''(\hat{u}-\hat{v})}{\|M''(\hat{u}-\hat{v})\|} = -2\|M''(\hat{u} - \hat{v})\|$ for $M'' = |M'|$

Stop thinking about inner products…

Start thinking about distances

Result of tricks

Using only a given term $H \in S$,

Construct a Hamiltonian $H_{\text{final}} = \sum_{uv} G^{uv}$,

Such that the minimum energy of a product state is

$$\min_{\rho=\rho_1\rho_2\dots\rho_n} \operatorname{Tr}[H_{\text{final}}\, \rho_1 \dots \rho_n]$$

$$= \min_{|\hat{u}|=1} \sum_{uv} -2\|M''(\hat{u}-\hat{v})\| \;=\; -2 \max_{|\hat{u}|=1} \sum_{uv} \|M''(\hat{u}-\hat{v})\|$$

Call this sufficiently “nice”, and try to prove such a function is NP-hard.

Vector Max-Cut Problems

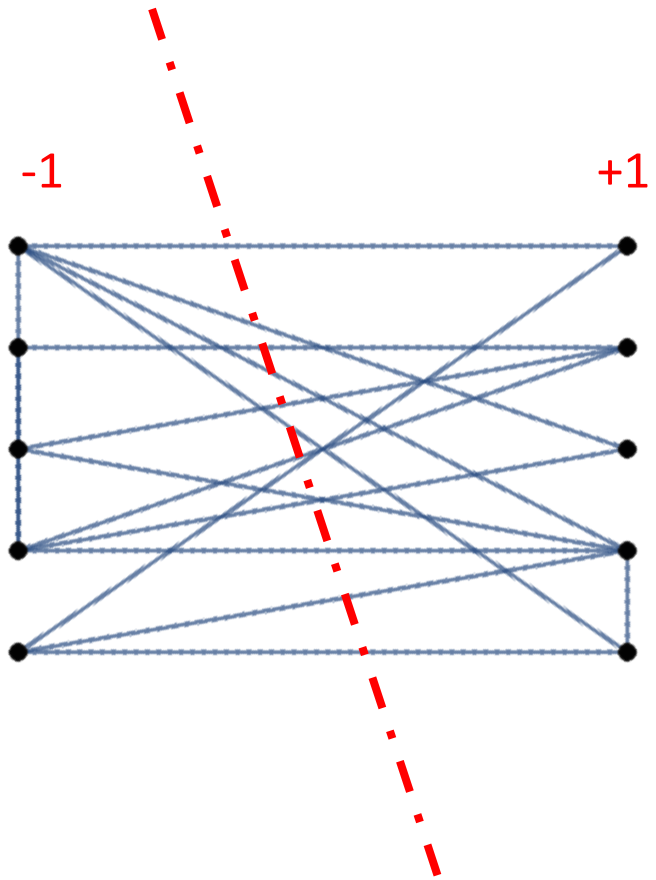
All classical TCS from here – no more quantum

Max-Cut

Max-Cut

$$\mathrm{MC}(G) = \frac{1}{2} \max_{\hat{\imath}=\pm 1} \sum_{ij\in E} 1 - \hat{\imath}\hat{\jmath}$$

$$= \frac{1}{2} \max_{\hat{\imath}=\pm 1} \sum_{ij\in E} |\hat{\imath} - \hat{\jmath}|$$



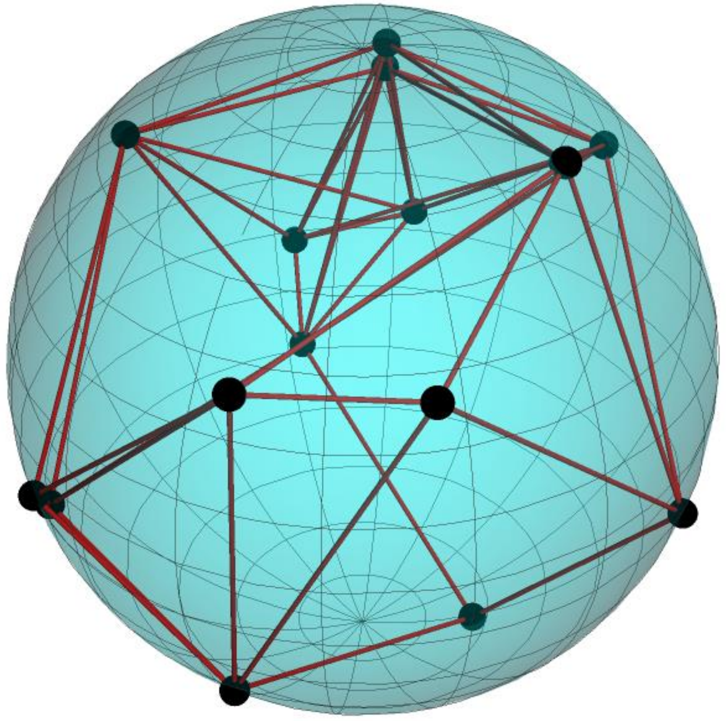
Vector Max-Cut

Max-Cut

$$\mathrm{MC}(G) = \frac{1}{2} \max_{\hat{\imath}=\pm 1} \sum_{ij\in E} 1 - \hat{\imath}\hat{\jmath}$$
$$= \frac{1}{2} \max_{\hat{\imath}=\pm 1} \sum_{ij\in E} |\hat{\imath} - \hat{\jmath}|$$

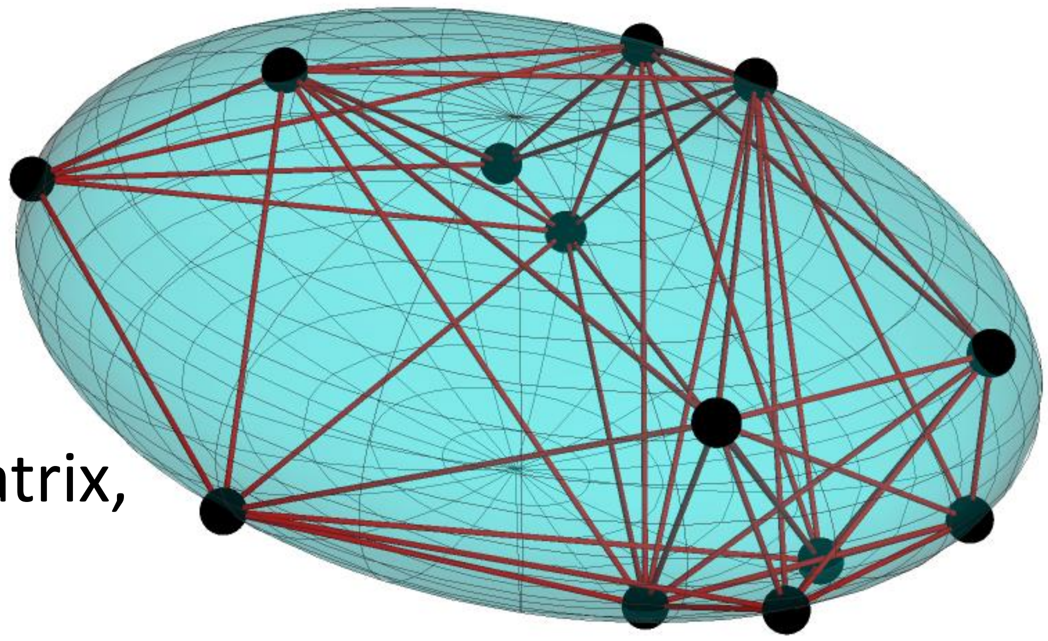
Vector Max-Cut

$$\mathrm{MC}_k(G) = \frac{1}{2} \max_{\hat{\imath}=S^{k-1}} \sum_{ij\in E} 1 - \hat{\imath}\hat{\jmath}$$
$$= \frac{1}{4} \max_{\hat{\imath}=S^{k-1}} \sum_{ij\in E} \|\hat{\imath} - \hat{\jmath}\|^2$$



Intuition: Embed a graph onto unit sphere S^{k-1} in $\mathbb{R}^k$ to maximize the sum of the squared distances

Stretched linear Vector Max-Cut



For $W = \begin{bmatrix} \alpha & & \\ & \beta & \\ & & \gamma \end{bmatrix}$ a fixed diagonal matrix,

given a graph $G = (V, E)$,

estimate

$$\mathrm{MC}_W^{\mathrm{L}}(G) = \frac{1}{2} \max_{\hat{u} \in S^{k-1}} \sum_{uv \in E} \|W\hat{u} - W\hat{v}\|$$

Intuition: Embed a graph onto ~~unit sphere~~ ellipsoid in $\mathbb{R}^k$ to maximize the sum of the ~~squared~~ distances

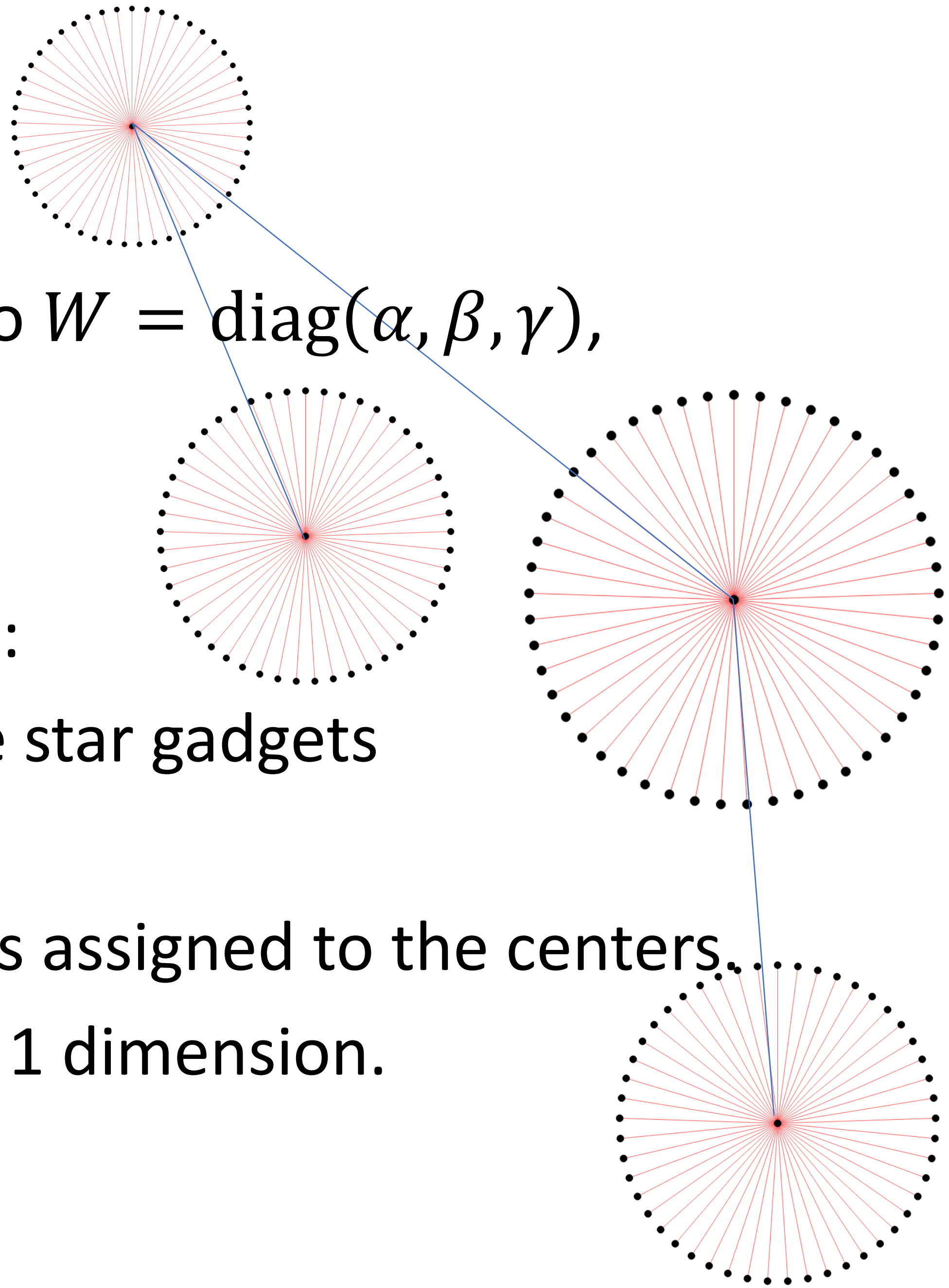
Stretched linear Vector Max-Cut is NP-hard

<u>**Theorem**</u>: For any fixed non-negative nonzero $W = \mathrm{diag}(\alpha, \beta, \gamma)$, $\mathrm{MC}_W^{\mathrm{L}}$ is NP-complete.

Earlier, we showed how to reduce an instance of $\mathrm{MC}_W^{\mathrm{L}}$ to S-prodLH.

✓So, this theorem will complete our main result: S-prodLH is NP-hard.

Proof sketch 1



Theorem: For any fixed non-negative nonzero $W = \mathrm{diag}(\alpha, \beta, \gamma)$, $\mathrm{MC}_W^{\mathrm{L}}$ is NP-complete.

If $\alpha > \beta, \gamma$, we can use a simple construction:

- Given graph G, construct G' by adding large star gadgets

The star gadgets *amplify* the length of vectors assigned to the centers.

To maximize the lengths, vectors must live in 1 dimension.

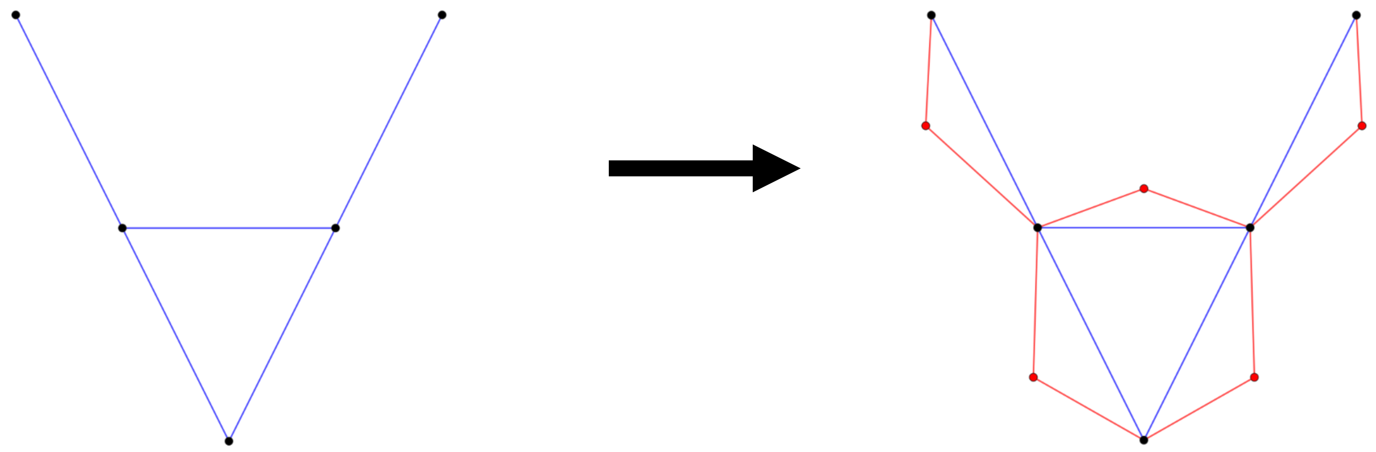
→ Reduction from standard Max-Cut

Proof sketch 2

Theorem: For any fixed non-negative nonzero $W = \mathrm{diag}(\alpha, \beta, \gamma)$, $\mathrm{MC}_W^{\mathrm{L}}$ is NP-complete.

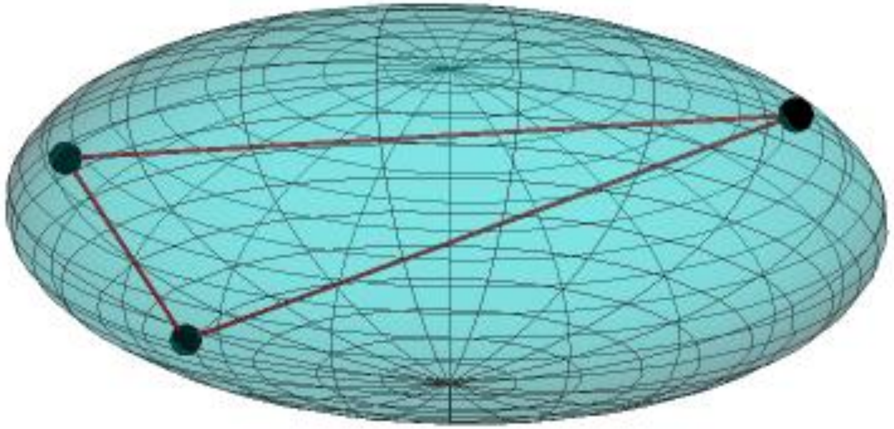
If $\alpha \ngtr \beta, \gamma$, we use a lengthier, but easy-to-analyze, construction.

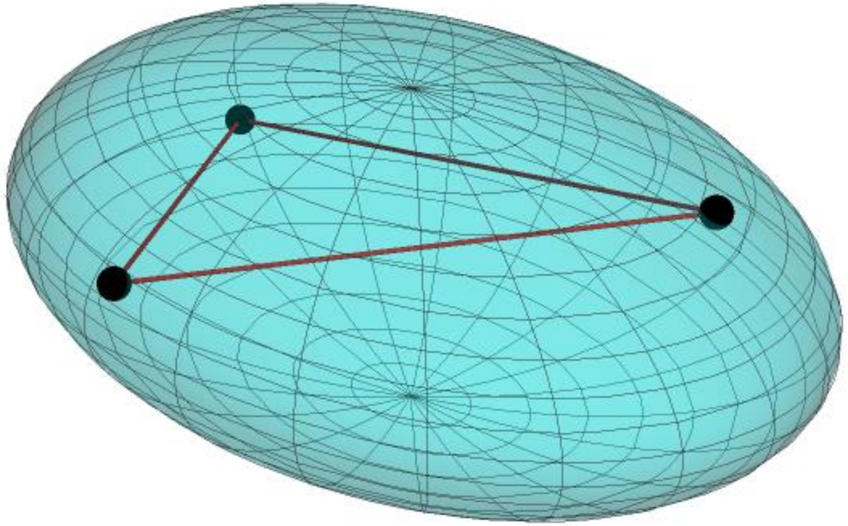
1. Given graph G, construct a new graph G' by replacing each edge with a 3-clique (triangle) gadget.



Proof sketch 2

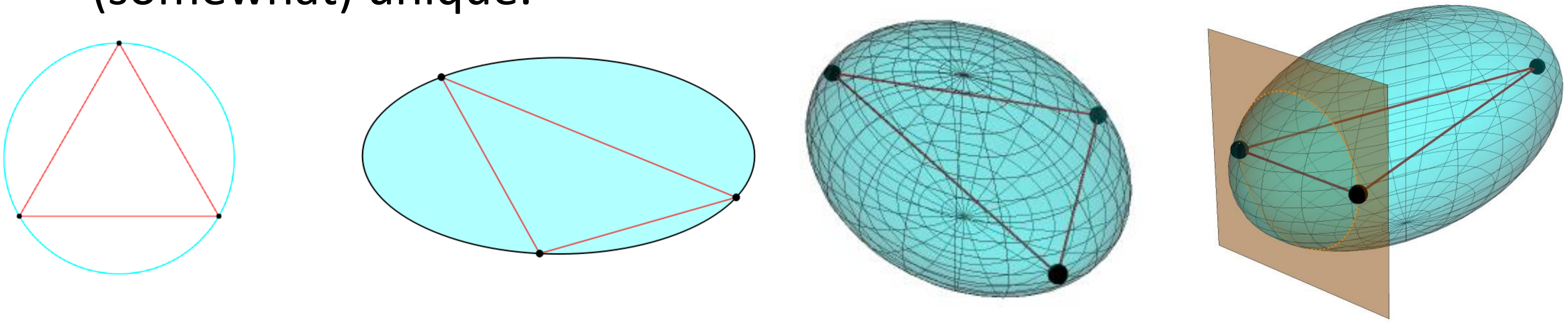
2. Observe that maximizing the total distance between the vectors in a 3-clique is equivalent to picking 3 points on an ellipsoid which inscribe a triangle with maximum perimeter.





Proof sketch 2

3. Use the fact that maximum perimeter inscribed triangles are (somewhat) unique.



Circle, Ellipse, Ellipsoid: fix any 1 point, there is exactly 1 max perimeter triangle.

Centroid: must fix **2** points to fully determine max perimeter triangle.

Proof sketch 2

4. Every 3-clique gadget shares a vertex with another 3-clique gadget.
   a) So, every gadget is assigned at least 1 vector shared with another gadget.
   b) Given 1 fixed point, there's a unique pair of other points giving maximum length…
5. Conclude that G' can maximally satisfy *every* 3-clique gadget iff the *same* set of 3 optimal vectors can be assigned to all 3-cliques.

The NP-complete **3-Coloring** problem reduces to the **Stretched linear Vector Max-Cut** problem. □

Summary of proof of main theorem

Proof summary

<u>Main Theorem</u>

For any fixed set of 2-qubit Hamiltonian terms S,

- if every matrix in S is 1-local then S-prodLH is in P,
- and otherwise S-prodLH is NP-complete.

✓ If every term is 1-local, then we can optimize the state of each qubit individually, so the problem is in P.

✓ prodLH is always contained in NP, using product states' concise classical descriptions $\mathrm{Tr}(H\rho) = \sum_{ij} \mathrm{Tr}(H_{ij}\, \rho_i \rho_j)$

❑ **To Do:** show if S contains a nontrivial 2-qubit term, then S-prodLH is NP-hard.

Proof summary

❑**To Do:** show if S contains a nontrivial 2-qubit term, then S-prodLH is NP-hard.

- Product state problems can be viewed as optimization over single-qubit Bloch vectors.
- Given an arbitrary non-trivial 2-qubit term, we construct gadgets to make the product state energy "nice".
- Call this new objective value **Stretched linear Vector Max Cut** ($\mathrm{MC}_W^{\mathrm{L}}$).
- Show $\mathrm{MC}_W^{\mathrm{L}}$ is NP-complete by reductions from 3-coloring or Max-Cut.

What's next?

1. Can we use the complexity of product state problems to suggest the general ground states of a class of Hamiltonians are *not* hard?

2. Classify S-prodLH with additional restrictions, e.g. only positive weights, spatial geometry?

3. Hamiltonian Constrained-Optimization problems, e.g. Quantum Vertex Cover

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On

Complexity Classification of Product State Problems for Local Hamiltonians

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Main Theorem: For any fixed set of 2-qubit Hamiltonian terms S,

- if every matrix in S is 1-local then S-prodLH is in P,
- and otherwise S-prodLH is NP-complete.

Corollary: For any fixed set of 2-qubit Hamiltonian terms S,

- the problem S-LH is at least NP-hard iff S-prodLH is NP-complete.
- the problem S-LH is in P iff S-prodLH is in P.